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Minimization of the sum of three linear fractional functions

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Abstract. In this paper, we will propose an efficient and reliable heuristic algorithm for minimizing and maximizing the sum of three linear fractional functions over a polytope. These problems are typical nonconvex minimization problems of practical as well as theoretical importance. This algorithm uses a primal-dual parametric simplex algorithm to solve a subproblem in which the value of one linear function is fixed. A subdivision scheme is employed in the space of this linear function to obtain an approximate optimal solution of the original problem. It turns out that this algorithm is much more efficient and usually generates a better solution than existing algorithms. Also, we will develop a similar algorithm for minimizing the product of three linear fractional functions.

Key words: Global optimization, Linear fractional function, Parametric simplex algorithm, Branch and bound method

1. Introduction

A number of special purpose algorithms have been developed in the last decade for solving a class of nonconvex minimization problems by exploiting their special structures. Readers are referred to such examples in recent books [7,12].

The purpose of this article is to propose an efficient and reliable heuristic algorithm for minimizing and maximizing the sum of $p(\leqslant 3)$ linear fractional functions over a polytope:

$$\left|\begin{array}{ll} \text{minimize} & \displaystyle\sum_{j=1}^{p} \frac{d_j^T x + d_{j0}}{c_j^T x + c_{j0}} \\ \text{subject to} & x \in X, \end{array}\right. \tag{1.1}$$

and

$$\left|\begin{array}{ll} \text{maximize} & \displaystyle\sum_{j=1}^{p} \frac{d_j^T x + d_{j0}}{c_j^T x + c_{j0}} \\ \text{subject to} & x \in X, \end{array}\right. \tag{1.2}$$

where $c_j, d_j \in R^n$, $c_{j0}, d_{j0} \in R^1$, $j = 1, \ldots, p$ and X is a set defined by a system of linear equalities and inequalities. Problems (1.1) and (1.2) will be called degree-p linear fractional programming problems. These problems have application in multi-stage stochastic shipping problem [1] and multi-objective bond portfolio optimization problems [9,16]. Also, there exist a number of mathematical studies on fractional programming problems; see [3,5,19].

Let us briefly discuss a classical example proposed by Almogy and Levin [1]. A ship must make an ordered tour around N ports, each of which has cargo available for shipping to the remaining ports to be visited. The amounts of available cargo at each port are independent random variables and the objective of the shipping company is to maximize the expected value of the net profit per unit time which is defined as a linear fractional function of the cargo to be loaded and unloaded.

Almogy and Levin [1] showed that the problem can be posed as (1.2) when $N = 2$. To see this, let $g_0(x)$ be a linear fractional function where x is a vector of cargo to be loaded at port 1 given the vector u of available cargo. Let us assume that the amount of available cargo at port 2 is either one of the vectors u^k, $k = 1, \ldots, K$. Let $g_k(x^k)$ be the linear fractional function associated with u^k. The expected value of the net profit per unit time is represented as follows:

$$g(x) + \sum_{k=1}^{K} p_k g_k(x^k),$$

where p_k is the probability of occurrence of u^k. The problem is therefore the maximization of degree $K + 1$ linear fractional functions under appropriate linear constraints on $(x, x^1, \ldots, x^K)$.

When $p = 1$, the problem (1.1) and (1.2) are well known linear fractional programming problems for which an efficient simplex-type algorithm has been developed by Charnes and Cooper [4] in 1962. In fact, the objective function is both quasi-convex and quasi-concave in this case. When $p = 2$, the objective function is no longer quasi-convex. However, we can construct a parametric simplex algorithm for (1.1) and (1.2) which can solve a large scale problem in an efficient manner [14]. Also, Falk et al. [6] proposed an alternative algorithm for these problems.

When $p \geqslant 3$, the problem is much more difficult. The only exact algorithms, to the authors' knowledge, are the one proposed by Konno and Yamashita [15] for problem (1.1) by adapting the algorithm for generalized convex multiplicative programming problems [11] and the one proposed by Falk et al. [6].

It has been proved in [11] that the problem (1.1) is equivalent to the following master problem:

$$\left| \begin{array}{ll} \text{minimize} & \sum_{j=1}^{p} \{\xi_j (d_j^T x + d_{j0})^2 + \eta_j (c_j^T x + c_{j0})^{-2}\} \\ \text{subject to} & x \in X, \\ & \xi_j \eta_j \geqslant 1,\ \xi_j \geqslant 0,\ \eta_j \geqslant 0,\ j = 1, \ldots, p, \end{array} \right. \tag{1.3}$$

provided the following condition:

$$\frac{(d_j^T x + d_{j0})}{(c_j^T x + c_{j0})} \geqslant 0, \quad \forall x \in X, \quad j = 1, \ldots, p, \tag{1.4}$$

is satisfied.

Let us denote $\xi = (\xi_1, \ldots, \xi_p)$, $\eta = (\eta_1, \ldots, \eta_p)$. When the value of (ξ, η) is fixed, the objective function is convex and hence we can calculate

$$G(\xi, \eta) = \min \left\{ \sum_{j=1}^{p} [\xi_j (d_j^T x + d_{j0})^2 + \eta_j (c_j^T x + c_{j0})^{-2}] \mid x \in X \right\}, \tag{1.5}$$

by standard algorithms for convex minimization problem. By using the fact that the function $G(\cdot, \cdot)$ is concave, we can construct an outer approximation algorithm as in [15] or a branch and bound algorithm [11] for calculating the global minimum of $G(\xi, \eta)$ over the convex region:

$$S = \{(\xi, \eta) \mid \xi_j \eta_j \geqslant 1,\ \xi_j \geqslant 0,\ \eta_j \geqslant 0,\ j = 1, \ldots, p\}. \tag{1.6}$$

It is reported in [15] that this outer approximation algorithm successfully solves the problem (1.1) up to $p = 4$. Unfortunately, the computation time sharply increases as p increases. For example, the computation time for $p = 3$ is about 50 times more than the parametric simplex method requires for $p = 2$. Also, this algorithm is not valid if the condition (1.4) is not satisfied.

The purpose of this paper is to propose an efficient heuristic algorithm for solving degree-3 linear fractional programming problems (1.1) and (1.2). This algorithm is an extension of the parametric simplex algorithm for degree-2 problems [14]. Unlike the generalized convex multiplicative programming approach, we do not have to assume the condition (1.4). This in turn implies that the new algorithm is applicable to problem (1.2) as well as (1.1).

In the next section, we will discuss the new algorithm in detail. Section 3 will be devoted to the results of the numerical experiments. It will be shown that the new algorithm is at least 10 times faster than the existing approach [15]. Also, this algorithm always generates a solution which is almost the same as the one calculated by an exact algorithm [15]. Finally, in Section 4, we will extend this algorithm to the minimization of the product of three linear fractional functions.

2. A parametric simplex algorithm for degree three linear fractional programming problems

Let us consider the following degree-3 linear fractional programming problems:

$$\left| \begin{array}{ll} \text{minimize} & \dfrac{d_1^T x + d_{10}}{c_1^T x + c_{10}} + \dfrac{d_1^2 x + d_{20}}{c_2^T x + c_{20}} + \dfrac{d_3^T x + d_{30}}{c_3^T x + c_{30}} \\ \text{subject to} & Ax = b, \quad x \geqslant 0, \end{array} \right. \tag{2.1}$$

$$\left|\begin{array}{ll}\text{maximize} & \dfrac{d_1^T x + d_{10}}{c_1^T x + c_{10}} + \dfrac{d_1^2 x + d_{20}}{c_2^T x + c_{20}} + \dfrac{d_3^T x + d_{30}}{c_3^T x + c_{30}} \\ \text{subject to} & Ax = b, \quad x \geqslant 0,\end{array}\right. \tag{2.2}$$

where $c_j, d_j \in R^n$, $c_{j0}, d_{j0} \in R^1$, $j = 1, \ldots, p$, and $A \in R^{m\times n}$, $b \in R^m$.

ASSUMPTION 1. The feasible region

$$X = \{x \in R^n \mid Ax = b,\ x \geqslant 0\} \tag{2.3}$$

is non-empty and bounded. □

ASSUMPTION 2. $c_j^T x + c_{j0} > 0$, $\forall x \in X$, $j = 1, 2, 3$. □

In The following, we will concentrate on the algorithm for solving problem (2.1) since the problem (2.2) can be converted to the problem (2.1) by reversing the sign of the objective function.

The first step for solving (2.1) is to introduce the so-called Charnes-Cooper transformation:

$$\begin{aligned} y_0 &= 1/(c_3^T x + c_{30}), \\ y &= x y_0. \end{aligned} \tag{2.4}$$

Then the problem (2.1) can be rewritten as follows:

$$\left|\begin{array}{ll}\text{minimize} & f(y, y_0) = \dfrac{d_1^T y + d_{10} y_0}{c_1^T y + c_{10} y_0} + \dfrac{d_2^T y + d_{20} y_0}{c_2^T y + c_{20} y_0} + d_3^T y + d_{30} y_0 \\ \text{subject to} & Ay - by_0 = 0, \\ & c_3^T y + c_{30} y_0 = 1, \\ & y \geqslant 0,\ y_0 \geqslant 0.\end{array}\right. \tag{2.5}$$

PROPOSITION 1. The feasible set Y of (2.5) is nonempty and bounded. Also, any $(y, y_0) \in Y$ satisfies $y_0 > 0$.

Proof. Let $x \in X$. Then $y = x/(c_3^T x + c_{30})$, $y_0 = 1/(c_3^T x + c_{30})$ gives a feasible solution of (2.5). Therefore, $Y \neq \phi$. If there exists $(y, 0) \in Y$, then we have $Ay = 0$, $y \geqslant 0$ which contradicts Assumption 1. Hence any $(y, y_0) \in Y$ satisfies $y_0 > 0$. If Y is unbounded, then there exists $(z, z_0) \neq (0, 0)$ satisfying the condition

$$Az - bz_0 = 0, \quad c_3^T z + c_{30} z_0 = 0, \quad z \geqslant 0,\ z_0 \geqslant 0.$$

It follows from Assumption 1 that $z_0 > 0$. Therefore, there exists a nonnegative vector x such that $Ax = b$, $c_3^T x + c_{30} = 0$, which is a contradiction to Assumption 2. □

PROPOSITION 2. The problem (2.5) has an optimal solution (y^*, y_0^*). Also, $x^* = y^*/y_0^*$ is an optimal solution of the problem (2.1).

Proof. Since $y/y_0 \in X$ for any $(y, y_0) \in Y$, we have

$$c_j^T y + c_{j0} y_0 = (c_j^T y/y_0 + c_{j0}) y_0 > 0, \quad j = 1, 2,$$

which implies that $(d_j^T y + d_{j0} y_0)/(c_j^T y + c_{j0} y_0)$ is pseudomonotonic on Y for $j = 1, 2$. Hence the objective function of (2.5) is continuous [2]. Therefore, from Proposition 1 above, the problem (2.5) has an optimal solution. The last statement of the Proposition follows directly from the definition (2.4). □

Let us define a new variable

$$\eta = d_3^T y + d_{30} y_0, \tag{2.6}$$

and let

$$\eta_{\max} = \max\{d_3^T y + d_{30} y_0 \mid (y, y_0) \in Y\}, \tag{2.7}$$

$$\eta_{\min} = \min\{d_3^T y + d_{30} y_0 \mid (y, y_0) \in Y\}. \tag{2.8}$$

Then the problem (2.5) is equivalent to

$$\left|\begin{array}{ll} \text{minimize} & \dfrac{d_1^T y + d_{10} y_0}{c_1^T y + c_{10} y_0} + \dfrac{d_2^T y + d_{20} y_0}{c_2^T y + c_{20} y_0} + \eta \\ \text{subject to} & (y, y_0) \in Y, \\ & d_3^T y + d_{30} y_0 = \eta, \\ & \eta_{\min} \leqslant \eta \leqslant \eta_{\max}. \end{array}\right. \tag{2.9}$$

We can assume without loss of generality that (d_3^T, d_{30}) is linearly independent of the rows of $(A, -b)$ and (c_3^T, c_{30}) since η is constant over Y if otherwise.

For fixed η, the problem (2.9) is a degree - 2 linear fractional programming problem:

$$P(\eta) \left|\begin{array}{ll} \text{minimize} & \dfrac{p_1^T z}{q_1^T z} + \dfrac{p_2^T z}{q_2^T z} + \eta \\ \text{subject to} & \tilde{A} z = b(\eta),\ z \geqslant 0, \end{array}\right. \tag{2.10}$$

where

$$z = \begin{pmatrix} y \\ y_0 \end{pmatrix}, \quad p_j = \begin{pmatrix} d_j \\ d_{j0} \end{pmatrix}, \quad q_j = \begin{pmatrix} c_j \\ c_{j0} \end{pmatrix},\ j = 1, 2. \tag{2.11}$$

$$\tilde{A} = \begin{pmatrix} A & -b \\ c_3^T & c_{30} \\ d_3^T & d_{30} \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ \eta \end{pmatrix}. \tag{2.12}$$

The feasible set of (2.1), denoted by $Y(\eta)$, is a subset of Y; hence, we have $q_j^T z > 0$, $\forall z \in Y(\eta)$. This enables one to apply Charnes-Cooper transformationl once again by defining

$$v_0 = 1/q_2^T z,$$
$$v = z v_0.$$

The problem (2.10) reduces to

$$P(\eta) \left| \begin{array}{ll} \text{minimize} & \dfrac{p_1^T v}{q_1^T v} + p_2^T v + \eta \\ \text{subject to} & \tilde{A}v - b(\eta)v_0 = 0, \\ & q_2^T v = 1, \\ & v \geqslant 0,\ v_0 \geqslant 0. \end{array} \right. \tag{2.13}$$

PROPOSITION 3. The feasible set of (2.14) is nonempty and bounded.

Proof. We can apply the same argument as the proof of Proposition 1. □

By introducing another auxiliary variable $\xi = q_1^T v$, the problem (2.14) can be represented as follows:

$$\left| \begin{array}{ll} \text{minimize} & \dfrac{1}{\xi} p_1^T v + p_2^T v + \eta \\ \text{subject to} & \tilde{A}v - b(\eta)v_0 = 0, \\ & q_2^T v = 1, \\ & q_1^T v = \xi, \\ & v \geqslant 0,\ v_0 \geqslant 0, \\ & \xi_{\min}(\eta) \leqslant \xi \leqslant_{\max} (\eta), \end{array} \right. \tag{2.14}$$

where

$$\xi_{\max}(\eta) = \max\{q_1^T v \mid \tilde{A}v - b(\eta)v_0 = 0,\ q_2^T v = 1,\ v \geqslant 0,\ v_0 \geqslant 0\}, \tag{2.15}$$

$$\xi_{\min}(\eta) = \min\{q_1^T v \mid \tilde{A}v - b(\eta)v_0 = 0,\ q_2^T v = 1,\ v \geqslant 0,\ v_0 \geqslant 0\}. \tag{2.16}$$

An optimal solution of (2.5) is obtained by solving $P(\eta)$ for all $\eta \in [\eta_{\min}, \eta_{\max}]$.

As discussed in detail in [14], $P(\eta)$ can be solved by a primal-dual parametric simplex method. In fact, let $B(\xi_0)$ be an optimal basis matrix of the problem (2.15) for some $\xi_0 \in [\xi_{\min}(\eta),\ \xi_{\max}(\eta)]$. Then we can find an interval $[\underline{\xi_0}, \overline{\xi_0}]$ in which $B(\xi_0)$ remains optimal by noting

(i) primal feasibility condition

$$B^{-1}(\xi_0) \begin{bmatrix} 0 \\ 1 \\ \xi \end{bmatrix} \geqslant 0 \Rightarrow \xi \in [\underline{\xi}^P,\ \overline{\xi}^P],$$

(ii) dual feasibility condition

$$\frac{1}{\xi}\,\overline{p}_{1N} + \overline{p}_{2N} \geqslant 0 \Rightarrow \xi \in [\underline{\xi}^{D}, \overline{\xi}^{D}],$$

where $\overline{p}_{1N}$ and $\overline{p}_{2N}$ are the reduced cost corresponding to p_1 and p_2.

Therefore, we obtain an interval

$$[\underline{\xi}, \overline{\xi}] \equiv [\underline{\xi}^{P}, \overline{\xi}^{P}] \cap [\underline{\xi}^{D}, \overline{\xi}^{D}],$$

in which the basis $B(\xi_0)$ remains optimal (Note that $\xi_0 \in [\underline{\xi}, \overline{\xi}]$). Therefore, we can generate a sequence of intervals in which a certain basis matrix remains optimal by applying a series of primal or dual simplex iterations. The value of the objective function is given by

$$\left[\frac{1}{\xi}\,p_{1B}^{T} + p_{2B}^{T}\right] \quad B^{-1}(\xi_0)\begin{bmatrix} 0 \\ 1 \\ \xi \end{bmatrix},$$

in each interval, where p_{1B} and p_{2B} represent the basic part of the vectors p_1 and p_2 associated with $B(\xi_0)$. Therefore, we can calculate the optimal value of $P(\eta)$ in an analytic manner; See [14] for details.

THEOREM 1. *Let $F(\eta)$ be the minimal value of the objective function of the problem $P(\eta)$. Then $F(\eta)$ is continuous on the interval* $(\eta_{\min}, \eta_{\max})$.

Proof. See Appendix A. □

The problem therefore reduces to the minimization of a continuous function $F(\cdot)$ of a single variable. However, this function need not be differentiable nor convex. Thus we will apply a primitive subdivision scheme to be described below.

Let us generate a finite number of grid points η_k, $k = 0, 1, \ldots, K$:

$$\eta_{\min} = \eta_0 < \eta_1 < \eta_2 < \ldots \eta_K = \eta_{\max},$$

and let f_k and x^k be, respectively, the optimal value and an optimal solution of $P(\eta_k)$. If $\varepsilon \equiv \max\{\eta_{k+1} - \eta_k \mid k = 0, 1, \ldots, K-1\}$ is small enough, then $f^* = \min[f_k \mid k = 0, 1, \ldots, K]$ and the associated solution x^* is expected to provide a good approximation of a global optimal solution of the problem (2.5).

ALGORITHM PD. PRIMAL-DUAL PARAMETRIC SIMPLEX ALGORITHM

Step 1. Calculate $\eta_{\min}$ and $\eta_{\max}$ by solving a pair of linear programs (2.7) and (2.8) and generate a finite number of grid points η_k, $k = 0, 1, \ldots, K$ satisfying the condition

$$\eta_{\min} = \eta_0 < \eta_1 < \eta_2 < \ldots < \eta_K = \eta_{\max}.$$

Step 2. For $k = 0, 1, \ldots, K$, solve $P(\eta_k)$ by primal dual simplex method. Let f_k be the minimal value of the objective function of $P(\eta_k)$.
Step 3. Let $f_k^* = \min\{f_k \mid k = 0, 1, \ldots, K\}$ and let z^* be an optimal solution of the problem $P(\eta_k^*)$.
Step 4. Stop. $x^* = y^*/y_0^*$ is an approximate optimal solution of (2.1). □

3. Computational results

We tested the primal-dual parametric simplex algorithm (PD) and compared it with the convex multiplicative programming algorithm (CM) proposed in [15].

We generated test problems of the following form:

$$\left|\begin{array}{l} \text{minimize} \ \displaystyle\sum_{j=1}^{3} \frac{d_j^T x + d_{j0}}{c_j^T x + c_{j0}} \\ \text{subject to} \ Ax \leqslant b,\ x \geqslant 0, \end{array}\right. \tag{3.1}$$

where the elements of $A \in R^{m\times n}$, c_j, $d \in R^n$, c_{j0}, $d_{j0} \in R^1$, $j = 1, 2, 3$ are randomly generated in the unit interval $[0.0, 1.0]$. Also, the elements of b are randomly generated in the interval $[0.1, 1.0]$ to satisfy the condition (1.4). Table 1 shows the computation time in seconds for CM algorithm and PD algorithm. The convergence condition $\varepsilon = 10^{-5}$ was chosen for CM algorithm. Also, we divided the interval $[\eta_{\min}, \eta_{\max}]$ into 100 subintervals of equal length in PD algorithm. We solved ten test problems of each size and listed the average computation time and its standard deviation. All test problems satisfy the condition $\eta_{\max} - \eta_{\min} \leqslant 1.0$, so that $\varepsilon < 0.01$ for 100 subdivisions.

Table I. Computation time

		(m,n)			
		(5,10)	(10,20)	(15,30)	(20,40)
CM	Ave. CPU time (sec)	6.76	47.01	221.58	(535.14)
	S.D. (sec)	5.54	38.93	102.22	(245.98)
PD	Ave. CPU time (sec)	1.65	11.27	46.21	152.81
	SD (sec)	0.25	1.96	6.98	23.24

Comparison was terminated at (m,n)=(20,40) since some test problems could not be solved within ten minutes by CM algorithm. (The data for (20,40) in the brackets show the average and standard deviation of eight test problems which were solved in ten minutes). We see from this that the computation time of PD algorithm is about five times less than that of CM. This is due to the fact that the number of pivots of PD algorithm is relatively stable. A remarkable difference is that the standard deviation of PD is at least ten times less than CM. We conclude from this that PD algorithm depends primarily on the size of the problem and the

Table II. Quality of the solution

	(m,n)			
	(5,10)	(10,20)	(15,30)	(20,40)
Ave. (f^*_{PD}/f^*_{CM})	0.9995	0.9709	0.9878	1.0005

Table III. Improvement of the objective value (%)

	(m,n)			
$K_1 \times K_2 \times K_3$	(5,10)	(10,20)	(15,30)	(20,40)
$40 \times 30 \times 30$	0.177	0.143	0.078	0.092

subdivision parameter K, while CM algorithm depends on both the size and the data.

Table 2 shows the quality of the calculated solutions. This table shows the ratio of the average of the magnitude of the optimal value f^*_{PD}, f^*_{CM} obtained by algorithm PD and CM. Surprisingly enough, the optimal value calculated by PD is usually better than those obtained by CM algorithm, sometimes by a factor of 2%. This means that PD algorithm with 100 subdivision detects a better solution than CM algorithm when $\varepsilon = 10^{-5}$. At the same time, this table shows that the quality of the solution of PD algorithm is almost the same or even a bit worse than those of CM algorithm as the size of the problem increases. To obtain a definite conclusion, however, we need to conduct more experiments on large scale problems.

Table 3 shows the computational result of PD for alternative subdivision schemes, where $(K_1 \times K_2 \times K_3)$ denotes the following three stage subdivision:

Stage 1. Subdivide $[\eta_{\min}, \eta_{\max}]$ into K_1 subintervals of the same length.

Stage 2. Divide the subintervals to the left and right of the best grid point obtained in Stage 1 into K_2 subintervals.

Stage 3. Divide the subintervals to the left and right of the best grid point obtained in Stage 2 into K_3 subintervals.

This scheme is based upon the observation that the minimal value $F(\eta)$ of $P(\eta)$ usually takes the form as depicted in Figure 1.

The calculated optimal solution was the same as $(100, 0, 0)$ for almost all problems and all subdivision schemes. Computation time of PD algorithm depends linearly on the number of subdivisions. The computation time of the parametric simplex algorithm for fixed value of η is no more than 10%–20% of the time required for calculating $\xi_{\min}(\eta)$. Therefore, the total amount of computation time for solving (2.9) for 100 subdivision scheme is more or less the same as solving 100 linear programs of the same size. Also, the computation time can be substantially reduced by using the fact that the starting solution for $\eta = \eta_{k+1}$ can be recovered

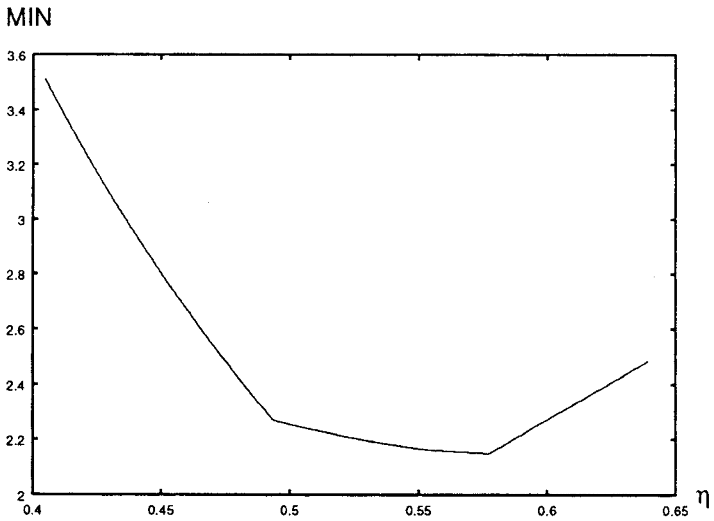

Figure 1. Minimal value of $P(\eta)$.

Table IV. Computation time for alternative subdivision

	(m,n)			
$K_1 \times K_2 \times K_3$	(5,10)	(10,20)	(15,30)	(20,40)
$100 \times 0 \times 0$	1.65	11.27	46.21	152.81
$30 \times 10 \times 10$	0.84	6.14	25.10	79.12
$20 \times 10 \times 10$	0.53	4.53	17.42	53.43

by dual simplex iteration (Note that the problem $P(\eta_{k+1})$ and $P(\eta_k)$ differs by a single constraint, namely $d_3^T y + d_{30} y_0 = \eta$).

Table 5 shows the computational result for lartger problems where (100, 0, 0) subdivision scheme was employed. These results are consistent with the observation above.

Let us note that test problems are all dense. Sparse problems may be solved much faster.

Table V. Computation time for larger problems

	(m,n)		
Ave. CPU time (sec)	22.8	118.5	284.3
SD (sec)	5.3	25.3	65.3

4. Extensions

The idea used for solving problem (2.1) can be applied to the minimization of the product of three linear fractional functions:

$$\left|\begin{array}{ll} \text{minimize} & \dfrac{d_1^T x + d_{10}}{c_1^T x + c_{10}} \cdot \dfrac{d_2^T x + d_{20}}{c_2^T x + c_{20}} \cdot \dfrac{d_3^T x + d_{30}}{c_3^T x + c_{30}} \\ \text{subject to} & x \in X, \end{array}\right. \tag{4.1}$$

if the following conditions are satisfied,

$$c_j^T x + c_{j0} > 0, \quad d_j^T x + d_{j0} \geqslant 0, \quad \forall x \in X, \quad j = 1, 2, 3. \tag{4.2}$$

The problem (4.1) is equivalent to

$$\left|\begin{array}{ll} \text{minimize} & \eta \, \dfrac{d_1^T x + d_{10}}{c_1^T x + c_{10}} \cdot \dfrac{d_2^T x + d_{20}}{c_2^T x + c_{20}} \\ \text{subject to} & x \in X, \\ & \dfrac{d_3 x + d_{30}}{c_3^T x + c_{30}} = \eta, \\ & \eta_{\min} \leqslant \eta \leqslant \eta_{\max}, \end{array}\right. \tag{4.3}$$

where

$$\eta_{\max} = \max\{(d_3^T x + d_{30})/(c_3^T x + c_{30}) \mid x \in X\}, \tag{4.4}$$

$$\eta_{\min} = \min\{(d_3^T x + d_{30})/(c_3^T x + c_{30}) \mid x \in X\}. \tag{4.5}$$

Let us define the subproblem

$$Q(\eta) \left|\begin{array}{ll} \text{minimize} & \dfrac{d_1^T x + d_{10}}{c_1^T x + c_{10}} \cdot \dfrac{d_2^T x + d_{20}}{c_2^T x + c_{20}} \\ \text{subject to} & (d_3 - \eta c_3)^T x = \eta c_{30} - d_{30}, \\ & x \in X, \end{array}\right. \tag{4.6}$$

for fixed $\eta \in (\eta_{\min}, \eta_{\max})$. As shown in [17], $Q(\eta)$ can be solved in an efficient manner by a branch and bound algorithm by noting the equivalence of $Q(\eta)$ and the problem

$$\left|\begin{array}{ll} \text{minimize} & \xi \dfrac{d_1^T x + d_{10}}{c_1^T x + c_{10}} + \dfrac{1}{\xi} \dfrac{d_2^T x + d_{20}}{c_2^T x + c_{20}} \\ \text{subject to} & (d_3 - \eta c_3)^T x = \eta c_{30} - d_{30}, \\ & \xi \geqslant 0, \quad x \in X, \end{array}\right. \tag{4.7}$$

under condition (4.2). When ξ is fixed, this problem can be solved by using the parametric simplex algorithm for degree-2 linear fractional programming problem. It has been demonstrated in [13] that the problem (4.7) can be solved by a branch and bound method. Therefore, we can construct a heuristic algorithm for solving (4.6) by discretizing the interval $[\eta_{\min}, \eta_{\max}]$ in an appropriate manner.

It is expected that this algorithm can be used as an efficient and reliable heuristic.

Appendix

A. Appendix: Proof of Theorem 1

The result can be proved by applying Theorem 1.17 of the recent book [18]. However, we will provide an elementary proof below for completeness.

Let us denote the feasible set of the problem $P(\eta)$ as follows:

$$Z(\eta) = \{z | Dz = d, \quad g^T z = \eta, \quad z \geqslant 0\},$$

where

$$D = \begin{pmatrix} A & -b \\ c_3^T & c_{30} \end{pmatrix}, \quad g = \begin{pmatrix} d_3 \\ d_{30} \end{pmatrix},$$

and g is linearly independent of rows of D. Let

$$F(\eta) = \min\{f(z) | z \in Z(\eta)\},$$

where

$$f(z) = \frac{p_1^T z}{q_1^T z} + \frac{p_2^T z}{q_2^T z} + \eta.$$

We will show that $F(\eta)$ is continuous on $(\eta_{\min}, \eta_{\max})$.

Let $\delta > 0$ and $\eta_1, \eta_2 \in (\eta_{\min}, \eta_{\max})$ such that $\eta_2 - \eta_1 = \varepsilon$, where $\varepsilon > 0$ is some constant. Also, let z^1, z^2 be, respectively the optimal solution of $P(\eta_1)$ and $P(\eta_2)$. We will first show that there exist a point $v^1 \in Z(\eta_2)$ such that $\|v^1 - z^1\| \leqslant \alpha\varepsilon$ where α is some positive constant. Let

$$z^* = \operatorname{argmax}\{g^T z | Dz = d, z \geqslant 0\},$$

and let

$$\alpha = \max\left\{ \frac{\|z^* - z\|}{g^T(z^* - z)} \middle| z \in Z(\eta_1) \right\}.$$

Let v^1 be the intersection of the line segment $[z^1, z^*]$ and $Z(\eta_2)$; see Figure A.1. Then $v^1 - z^1 = c(z^* - z^1)$ for some constant c. Therefore

$$\frac{\|v^1 - z^1\|}{g^T(v^1 - z^1)} = \frac{\|z^* - z^1\|}{g^T(z^* - z^1)} \leqslant \alpha.$$

Hence

$$\|v^1 - z^1\| \alpha g^T(v^1 - z^1) = \alpha\varepsilon,$$

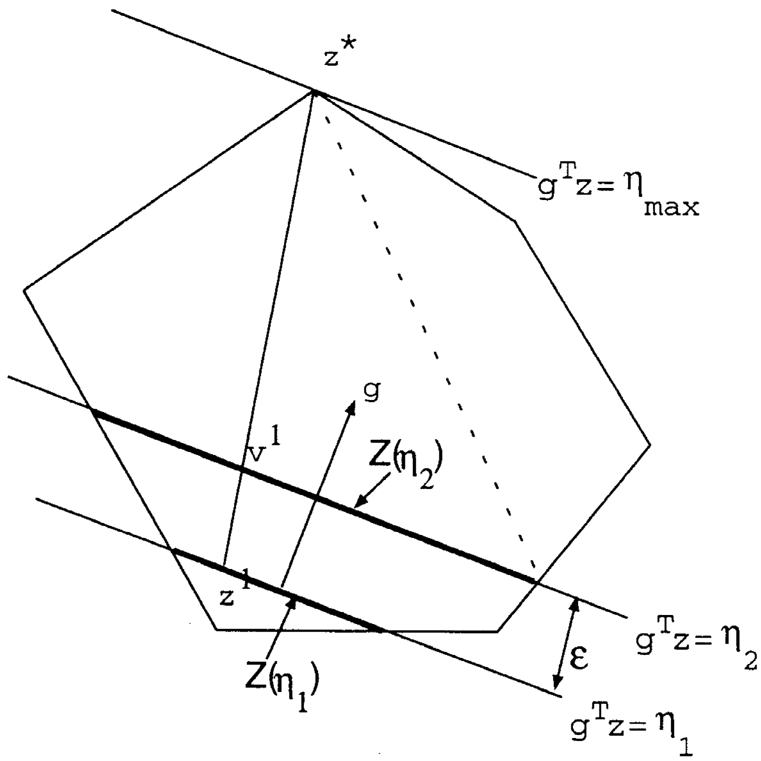

Figure A.1.

by noting that $g^T v^1 = \eta_2$, $g^T z^1 = \eta_1$ and $\eta_2 - \eta_1 = \varepsilon$. Similarly, there exists $v^2 \in Z(\eta_1)$ such that $\|v^2 - z^2\| \leqslant \alpha\varepsilon$.

By continuity of $f(z)$, we have

$$|f(v^1) - f(z^1)| \leqslant \delta, \quad |f(v^2) - f(z^2)| \leqslant \delta,$$

for small enough ε. By definition,

$$f(z^1) \leqslant f(v^2), \quad f(z^2) \leqslant f(v^1).$$

Hence,

$$\begin{aligned} f(z^1) - f(z^2) &= f(z^1) - f(v^2) + f(v^2) - f(z^2) \\ &\leqslant f(v^2) - f(z^2) < \delta, \\ f(z^1) - f(z^2) &= f(z^1) - f(v^1) + f(v^1) - f(z^2) \\ &\geqslant f(z^1) - f(v^1) \geqslant -\delta. \end{aligned}$$

Therefore, we have

$$|F(\eta_1) - F(\eta_2)| < |f(z^1) - f(z^2)| < \delta,$$

as desired. □

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